Solution of Dirichlet Problems by the Exodus Method

Matthew N. O. Sadiku, *Senior Member, IEEE*, and David T. Hunt

***Abstract*—The paper proposes the application of the Exodus method to Dirichlet problems in rectangular and axisymmetric solution regions. The stochastic technique is illustrated with specific practical applications to the solution of Laplace's equation. Although the method is probabilistic in its approach, it is not subject randomness as other Monte Carlo techniques because it does not involve the use of a pseudo-random generation subroutine. The method provides a more accurate solution in less amount of time compared with the fixed random walk. It is also found that the accuracy of the Exodus method is comparable to that of the finite difference method.**

I. Introduction

The MONTE CARLO technique is essentially a means of estimating expected values, and hence is a form of numerical quadrature. Although the technique can be applied to simple processes and estimating multidimensional integrals, the technique has been applied with success by various workers to solve potential problems [1]–[5]. Besides the fact that the Monte Carlo methods (MCM's) do not require input data, they are conceptually easier to understand and program compared with other numerical techniques.

There are two kinds of Monte Carlo methods. One type of MCM's requires the use of random numbers. This includes the *fixed random walk* and the *floating random walk*. A common feature of these MCM's is the idea of randomness of walk. The other type of MCM's does not require using random numbers. A typical example of this type is the so-called *Exodus method*, first suggested in [6] and applied to heat problems. While the random walk MCM's are popular in the electromagnetic community, the Exodus method is not so familiar. Although the method is probabilistic in its approach, it is not subject to randomness as other Monte Carlo techniques because it does not involve the use of a pseudo-random generation subroutine. This makes the solution independent of the computing facilities. Also the method provides results with the same degree of accuracy as those obtained using the finite difference method or the regular random walk Monte Carlo methods.

A simple introduction to Monte Carlo techniques, including the Exodus method, is presented in [7]. In this paper, we apply the Exodus method to Dirichlet problems in rectangular and axisymmetric solution regions. The technique is illustrated with specific practical applications to the solution of Laplace's equation. The method gives the "exact" solution in the sense that the main source of error is in the estimation of the transition probabilities which are calculated in such a way that randomness is avoided.

Manuscript received March 5, 1991; revised June 25, 1991.
The authors are with the Department of Electrical Engineering, Temple University, Philadelphia, PA 19122.
IEEE Log Number 9104478.

II. Theory

In this section we briefly present the theoretical background for the Exodus method. We specifically apply the method to Dirichlet problems in both rectangular and axisymmetric solution regions.

To apply the Exodus method in finding the solution of a potential problem usually involves the following three steps:

1) We first obtain the random walk probabilities from the finite difference equivalent of the partial differential equation describing the problem.
2) The Exodus method is used along with the random walk probabilities in calculating the transition probabilities.
3) The potential at the point of interest is finally obtained using the transition probabilities and the boundary conditions.

As with other MCM's, the major disadvantage of the Exodus method is that it only permits calculating the potential at one point at a time. With fast computing facilities, this drawback is not a problem if the potential at few points is needed and accurate solutions are necessary.

A. Rectangular Solution Region

Suppose the Exodus method is to be applied in solving Laplace's equation

$$\nabla^2 V = 0 \text{ in region } R \tag{1}$$

subject to Dirichlet boundary condition

$$V = V_b \text{ on boundary } B. \tag{2}$$

We begin by dividing the rectangular solution region R into a mesh and derive the finite difference equivalent. Assuming a mesh with $\Delta x = \Delta y = \Delta$, the finite difference equivalent of (1) is [8]

$$V(x, y) = p_{x+} V(x + \Delta, y) + p_{x-} V(x - \Delta, y) + p_{y+} V(x, y + \Delta) + p_{y-} V(x, y - \Delta) \tag{3}$$

where

$$p_{x+} = p_{x-} = p_{y+} = p_{y-} = \tfrac{1}{4}. \tag{4}$$

A probabilistic interpretation to (3) is that a random walking particle at an arbitrary point (x, y) in R has probabilities p_{x+}, p_{x-}, p_{y+}, and p_{y-} of moving from (x, y) to the neighboring points $(x + \Delta, y)$, $(x - \Delta, y)$, $(x, y + \Delta)$, and $(x, y - \Delta)$ respectively. Henceforth, p_{x+}, p_{x-}, p_{y+}, and p_{y-} will be referred to as the *random walk probabilities*.

It is also important to consider an extension of the MCM for the solution of problems with discrete homogeneities, i.e., homogeneous media separated by interfaces. Consider the interface along y = constant plane as shown in Fig. 1. The finite difference equivalent of the boundary condition $D_{1n} = D_{2n}$ at the interface is obtained by applying $\oint \boldsymbol{D} \cdot d\boldsymbol{S} = 0$ to the interface. The result is [9]

$$V_o = p_{x+} V_1 + p_{x-} V_2 + p_{y+} V_3 + p_{y-} V_4 \tag{5}$$

where

$$p_{x+} = p_{x-} = \frac{1}{4}, \quad p_{y+} = \frac{\epsilon_1}{2(\epsilon_1 + \epsilon_2)},$$

$$p_{y-} = \frac{\epsilon_2}{2(\epsilon_1 + \epsilon_2)}. \tag{6}$$

An interface along x = constant plane can be treated in a similar manner.

On a line of symmetry, the condition $\partial V/\partial n = 0$ must be imposed. If the line of symmetry is along the y-axis as in Fig. 2(a):

$$V_o = p_{x+} V_1 + p_{y+} V_3 + p_{y-} V_4 \tag{7}$$

where

$$p_{x+} = \tfrac{1}{2} \quad p_{y+} = p_{y-} = \tfrac{1}{4}. \tag{8}$$

The line of symmetry along the x-axis, shown in Fig. 2(b), is treated similarly.

B. Axisymmetric Solution Region

For $V = V(\rho, z)$, (1) becomes

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{9}$$

For ease of computation, we assume a square grid so that the step sizes along ρ and z coordinates are equal, i.e., $\Delta\rho = \Delta z = \Delta$. The finite difference approximation for $\rho \neq 0$ is [10]

$$V(\rho, z) = p_{\rho+} V(\rho + \Delta, z) + p_{\rho-} V(\rho - \Delta, z) + p_{z+} V(\rho, z + \Delta) + p_{z-} V(\rho, z - \Delta) \tag{10}$$

where

$$p_{z+} = p_{z-} = \frac{1}{4} \tag{11a}$$

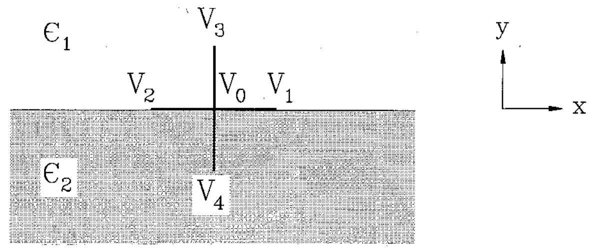


Fig. 1. Interface between media of dielectric permittivities ϵ_1 and ϵ_2.

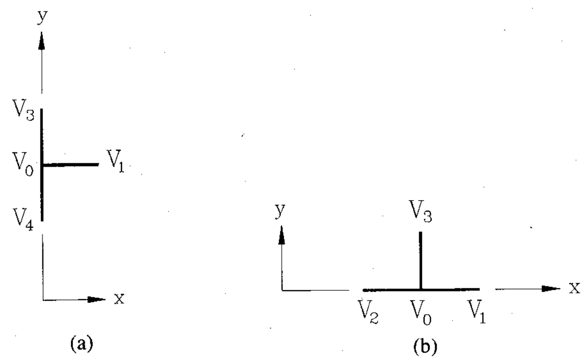


Fig. 2. Satisfying symmetry conditions: (a) $\partial V/\partial x = 0$, (b) $\partial V/\partial y = 0$.

$$p_{\rho+} = \frac{1}{4} + \frac{\Delta}{8\rho} \tag{11b}$$

$$p_{\rho-} = \frac{1}{4} - \frac{\Delta}{8\rho}. \tag{11c}$$

Note that the random walk probabilities satisfy

$$p_{z+} + p_{z+} + p_{\rho+} + p_{\rho-} = 1. \tag{12}$$

Equations (10) and (11) do not apply when $\rho = 0$. Since $\partial V/\partial \rho = 0$ at $\rho = 0$, applying L'Hopital's rule yields

$$\lim_{\rho \to 0} \frac{1}{\rho}\frac{\partial V}{\partial \rho} = \frac{\partial^2 V}{\partial \rho^2}. \tag{13}$$

Hence at $\rho = 0$, Laplace's equation becomes

$$2\frac{\partial^2 V}{\partial \rho^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{14}$$

and the finite difference equivalent is

$$V(0, z) = p_{\rho+} V(\Delta, z) + p_{z+} V(0, z + \Delta) + p_{z-} V(0, z - \Delta) \tag{15}$$

where

$$p_{z+} = p_{z-} = \tfrac{1}{6}, \quad p_{\rho+} = \tfrac{4}{6}, \quad p_{\rho-} = 0. \tag{16}$$

For the z = constant interface, the boundary condition $D_{1n} = D_{2n}$ or $\epsilon_1 \partial V_1/\partial z = \epsilon_2 \partial V_2/\partial z$ leads to random walk probabilities:

$$p_{z+} = \frac{\epsilon_1}{2(\epsilon_1 + \epsilon_2)}$$

$$p_{z-} = \frac{\epsilon_2}{2(\epsilon_1 + \epsilon_2)}, \tag{17}$$

while $p_{\rho -}$ and $p_{\rho +}$ remain as in (11). For ρ = constant interface, the boundary condition $(\epsilon_1/\rho)\partial V_1/\partial \rho = (\epsilon_2/\rho)\partial V_2/\partial \rho$ leads to

$$p_{\rho +} = \frac{\epsilon_1}{2(\epsilon_1 + \epsilon_2)}\left(1 + \frac{\Delta}{2\rho}\right)$$

$$p_{\rho -} = \frac{\epsilon_2}{2(\epsilon_1 + \epsilon_2)}\left(1 - \frac{\Delta}{2\rho}\right), \tag{18}$$

while p_{z-} and p_{z+} remain as in (11).

C. Exodus Method

Suppose for concreteness we are interested in solving the problem defined in (1) and (2), and the potential at a specific point (x_o, y_o) is to be determined. We define the *transition probability* p_k as the probability that a random walk starting at the point of interest (x_o, y_o) in R ends at a boundary point (x_k, y_k) with prescribed potential $V_b(k)$, i.e.,

$$p_k = \text{Prob}\,(x_o, y_o \rightarrow x_k, y_k). \tag{19}$$

If there are M boundary or fixed nodes (excluding the corner points since a random walk never terminates at those points), the potential at the starting point (x_o, y_o) of the random walks is

$$V(x_o, y_o) = \sum_{k=1}^{M} p_k V_b(k). \tag{20}$$

If m is the number of different boundary potentials, (20) can be simplified to

$$V(x_o, y_o) = \sum_{k=1}^{m} p_k V_b(k), \tag{21}$$

where p_k in this case is the probability that a random walk terminates on boundary k. Since $V_b(k)$ is specified, our problem is reduced to finding p_k. It is evident from (21) that the value of $V(x_o, y_o)$ would be "exact" if only the transition probabilities p_k are calculated exactly. The values of p_k can be obtained analytically using an expansion technique described in [11]. But this approach is limited to homogeneous rectangular solution regions. For inhomogeneous or nonrectangular regions, we must resort to some numerical technique. The Exodus method offers a numerical means of finding p_k.

To apply the Exodus method, let $P(i, j)$ be the number of particles at point (i, j) in R. We begin the application of the Exodus method by setting $P(i, j) = 0$ at all nodes (both fixed and free) except at free node (x_o, y_o) where $P(i, j)$ assumes a large number N (say, $N = 10^6$ or more). In other words, we introduce a large number of particles at (x_o, y_o) to start with. By scanning the mesh as is usually done in finite difference analysis, we dispatch the particles at each free node to its neighboring nodes according to the random walk probabilities p_{x+}, p_{x-}, p_{y+}, and p_{y-} as illustrated in Fig. 3. Note that in Fig. (3b), new $P(i, j) = 0$ at the node, while old $P(i, j)$ is shared among the neighboring nodes. At the end of each iteration (i.e. scanning of the free nodes in R as illustrated in Fig. 3), we record the number of particles that have reached the boundary (i.e. the fixed nodes), where the particles are absorbed. We keep scanning the mesh in a manner similar to the iterative process applied in finite difference solution [8] until a set number of particles (say 99.99% of N) have reached the boundary. If N_k is the number of particles that reached boundary k, we calculate

$$p_k = \frac{N_k}{N}. \tag{22}$$

Hence (21) can be written as

$$V(x_o, y_o) = \frac{\sum_{k=1}^{m} N_k V_b(k)}{N}. \tag{23}$$

Thus the problem is reduced to just finding N_k using the Exodus method, given N and $V_p(k)$. We notice that if $N \rightarrow \infty$, $\Delta \rightarrow 0$, and all the particles were allowed to reach the boundary points, the values of p_k and consequently $V(x_o, y_o)$ would be exact. It is easier to approach this exact solution using Exodus method than other MCM's and perhaps other numerical techniques such as finite difference and finite element methods. This fact will be demonstrated with examples in the next section.

For an axisymmetric solution region, we find the transition probability $p_k = \text{Prob}\,(\rho_o, z_o \rightarrow \rho_k, z_k)$ in the same way except that at each node we use random walk probabilities p_{z+}, p_{z+}, $p_{\rho +}$, and $p_{\rho -}$ in dispatching the particles.

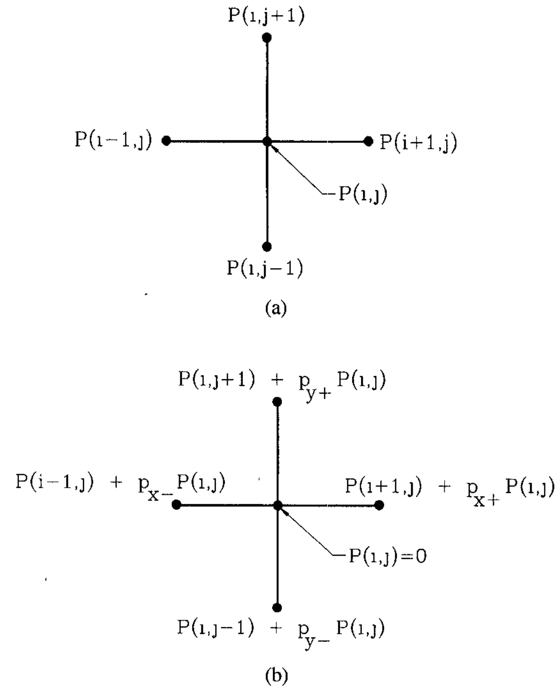


Fig. 3. (a) Before the particles at (i, j) are dipatched, (b) After the particles at (i, j) are dispatched.

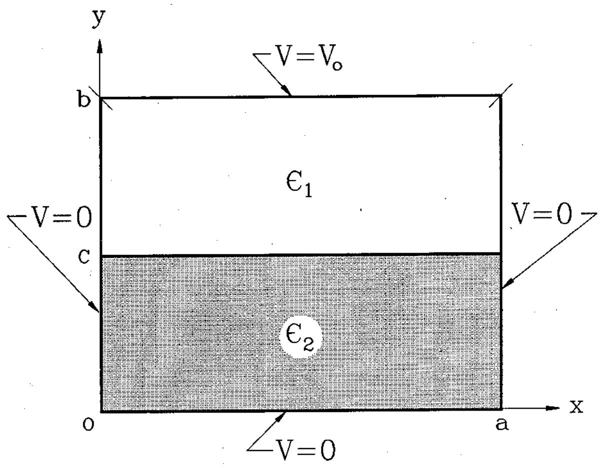


Fig. 4. Potential system for Example 1.

III. Typical Examples

We shall illustrate the application of the Exodus method by means of three numerical examples. The first example has an analytic solution so that the accuracy and validity of the Exodus method can be checked. The results from the two other examples are compared with the corresponding fixed random walk MCM and finite difference solutions.

Example 1: Consider the potential problem shown in Fig. 4. The potentials at $x = 0$, $x = a$, and $y = 0$ sides are zero while the potential at $y = b$ side is V_o. The analytic solution to this problem using series expansion technique [12], [13] is

$$V = \begin{cases} \sum_{k=1}^{\infty} \sin \beta x [a_n \sinh \beta y + b_n \cosh \beta y], & 0 \leq y \leq c \\ \sum_{k=1}^{\infty} c_n \sin \beta x \sinh \beta y, & c \leq y \leq b \end{cases} \tag{24}$$

where

$$\beta = \frac{n\pi}{a}, \quad n = 2k - 1,$$

$$a_n = 4V_o[\epsilon_1 \tanh \beta c - \epsilon_2 \coth \beta c]/d_n,$$

$$b_n = 4V_o(\epsilon_2 - \epsilon_1)/d_n,$$

$$c_n = 4V_o[\epsilon_1 \tanh \beta c - \epsilon_2 \coth \beta c + (\epsilon_2 - \epsilon_1) \coth \beta c]/d_n,$$

$$d_n = n\pi \sinh \beta b [\epsilon_1 \tanh \beta c - \epsilon_2 \coth \beta c + (\epsilon_2 - \epsilon_1) \coth \beta b]. \tag{25}$$

Typically, values

$$V_o = 100, \quad \epsilon_1 = \epsilon_o, \quad \epsilon_2 = 2.25\epsilon_o,$$

$$a = 3.0, \quad b = 2.0, \quad c = 1.0$$

were used in all calculations. The potentials were calculated at five typical points using the Exodus method, the fixed random walk Monte Carlo method, and the analytic solution. The number of particles, N, was taken as 10^7 for the Exodus method and the step size $\Delta = 0.05$ was used. For the fixed random walk method, $\Delta = 0.05$ and 2000 walks were used. It was noted that 2000 walks were sufficient for the random walk solutions to converge. The results are displayed in Table I. In the table, δ is the error estimate, which is obtained by repeating each calculation five times and using statistical formulas provided in [14]. It should be noted from the table that the results of the Exodus method agree to four significant places with the exact solution. Thus the Exodus method is more accurate than the random walk technique. It should also be noted that the Exodus method does not require the use of a random number routine and also the need of calculating the error estimate. The Exodus method, therefore, takes less computation time than the random walk method.

TABLE I
Results of Example 1

x	y	Exodus Method V	Fixed Random Walk ($V \pm \delta$)	Finite Difference V	Exact Solution V
0.5	1.0	13.41	13.40 ± 1.113	13.16	13.41
1.0	1.0	21.13	20.85 ± 1.612	20.74	21.13
1.5	1.0	23.43	23.58 ± 1.2129	22.99	23.43
1.5	0.5	10.52	10.13 ± 0.8789	10.21	10.51
1.5	1.5	59.36	58.89 ± 2.1382	59.06	59.34

To be able to check the results of the next two examples, which do not have exact solutions, we decided to solve this example using the finite difference method. The finite difference results are shown in Table I for the same step size $\Delta = 0.05$ and 1000 iterations. It was observed that 1000 iterations were sufficient for the finite difference solutions. It should be noted from Table I that the Exodus method gives a solution as accurate as the finite difference method.

Example 2: Consider the potential problem shown in Fig. 5. The potentials at $x = 0$, $x = w$, and $y = 0$ sides are zero while the potential at $y = h$ side is V_o. The problem is solved in [4] using the fixed random walk technique. Typically, values

$$V_o = 100, \quad \epsilon_1 = \epsilon_o, \quad \epsilon_2 = 3\epsilon_o, \quad a = b = 0.5,$$

$$h = w = 1.0$$

were used in all calculations. This problem was solved using the Exodus method with $\Delta = 0.05$ and $N = 10^7$. At the corner point $(x, y) = (a, b)$, (6) does not apply. It can be shown by applying $\oint \boldsymbol{D} \cdot d\boldsymbol{S} = 0$ that at this point

$$p_{x+} = p_{y+} = \frac{\epsilon_1}{3\epsilon_1 + \epsilon_2}, \quad p_{x-} = p_{y-} = \frac{(\epsilon_1 + \epsilon_2)}{2(3\epsilon_1 + \epsilon_2)}.$$

Since the problem has no exact solution, we compare the result with those obtained using the fixed random walk method with $\Delta = 0.05$ and 2000 walks and finite different method with $\Delta = 0.05$ and 500 iterations. The 2000 walks

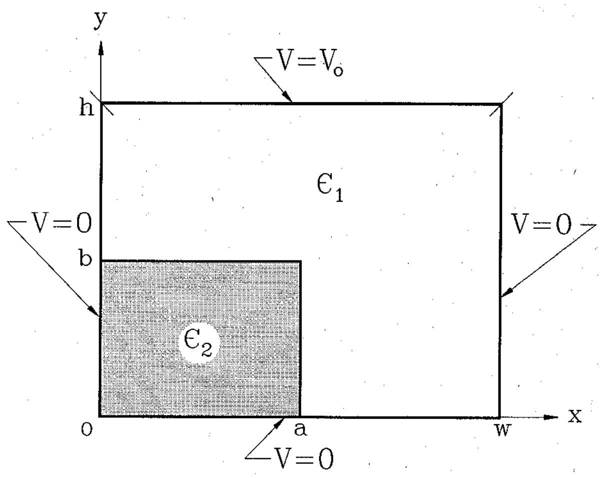


Fig. 5. Potential system for Example 2.

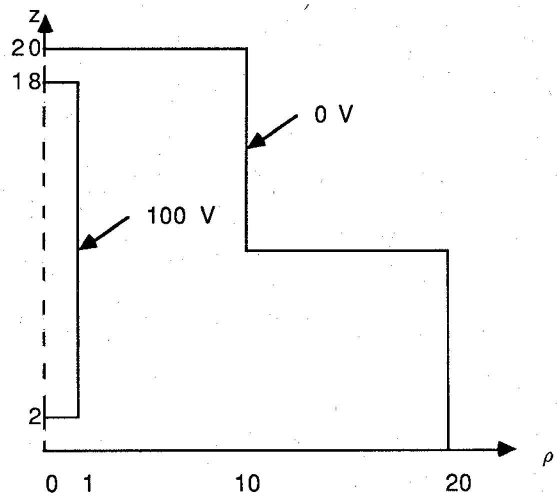


Fig. 6. An electrostatic particle focusing system; for Example 3 (all dimensions are in cm).

TABLE II
RESULTS OF EXAMPLE 2

x	y	Exodus Method V	Fixed Random Walk ($V \pm \delta$)	Finite Difference V
0.25	0.5	10.269	9.7951 ± 1.0593	10.166
0.5	0.5	16.667	16.602 ± 1.0865	16.576
0.75	0.5	15.931	15.872 ± 1.0247	15.887
0.5	0.75	51.931	50.775 ± 2.0394	50.928
0.5	0.25	6.2163	6.1069 ± 0.9585	6.1772

TABLE III
RESULTS OF EXAMPLE 3

ρ	z	Exodus V	Fixed Random Walk ($V \pm \delta$)	Finite Difference V
5	18	11.438	10.75 ± 0.6345	11.474
5	10	27.816	25.98 ± 1.777	27.869
5	2	12.179	11.44 ± 0.8402	12.128
10	2	2.3523	2.48 ± 0.5528	2.3421
15	2	0.38423	0.49 ± 0.2648	0.3965

and 500 iterations were enough for the convergence of random walk and finite difference solutions respectively. Table II presents the results for five typical points. As evident from the table, the Exodus method provides a more accurate solution in less amount of time compared with the fixed random walk. It is felt that the solution from the Exodus method is as accurate as that from finite difference method from the experience gained in Example 1.

Example 3: The last example is an axisymmetric problem shown in Fig. 6. This particular example is actually a prototype of an electrostatic particle focusing system employed in a recoil-mass time-of-flight spectrometer. It is intractible by analytic methods, yet presents no real difficulties for a Monte Carlo treatment. The fixed random walk solution to this problem is presented in [5]. Using the Exodus method, the potentials were calculated at five typical points with $\Delta = 0.2$ and $N = 10^7$. The results are compared with both the fixed random walk and the finite difference results in Table III. For the finite difference calculations, $\Delta = 0.25$ and $\Delta = 0.5$ gave nearly the same results for 1000 iterations. Again, it is evident from Table III that the results of the Exodus and finite difference methods closely agree. Also, it is noticed that the Exodus

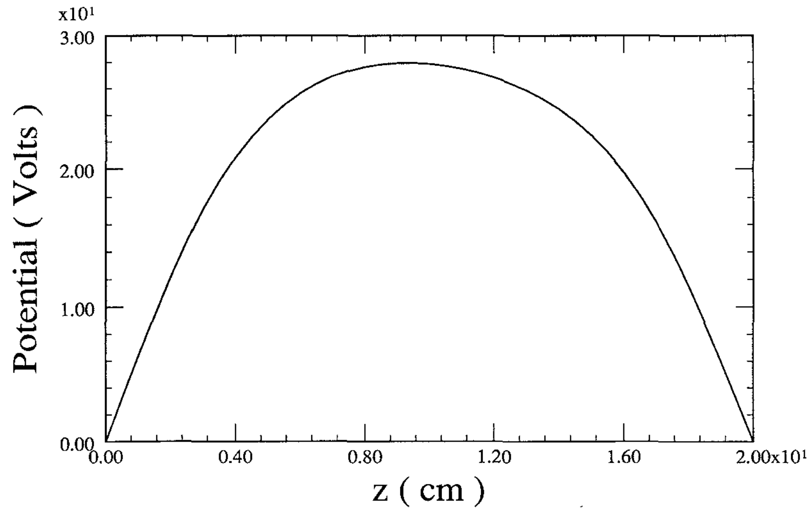


Fig. 7. Potential distribution along $\rho = 5$ cm, $0 \leq z \leq 20$ cm; for Example 3.

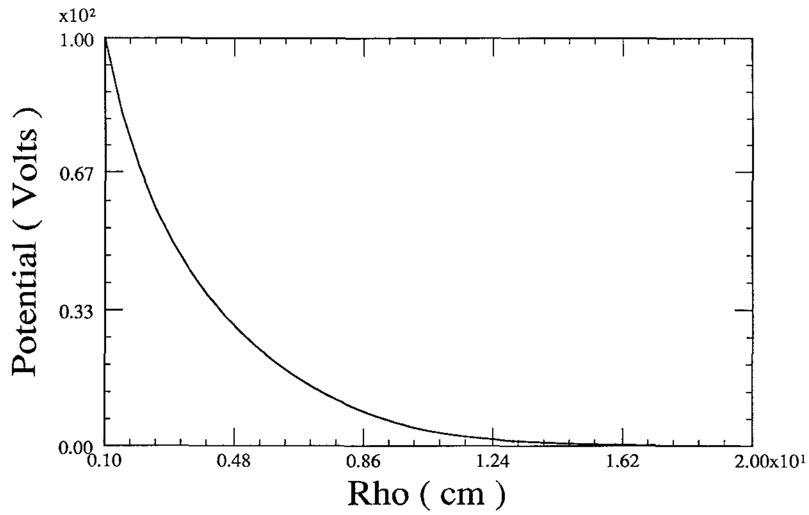


Fig. 8. Potential distribution along $z = 8$ cm, $0 \leq \rho \leq 20$ cm; for Example 3.

method is more accurate than the fixed random walk. Figs. 7 and 8 show the potential distribution (obtained using the Exodus method) along $\rho = 5$ cm, $0 \leq z \leq 20$ cm and $z = 8$ cm, $1 \leq \rho \leq 20$ cm respectively.

IV. Conclusions

The Exodus method provides a relatively straightforward means of solving Dirichlet problems. The method has been illustrated with three typical problems in rectangular and axisymmetric solution regions. The method provides a more accurate solution in less amount of time compared with the fixed random walk. Although the method is probabilistic in its approach, it is not subject to randomness as other Monte Carlo techniques because it does not involve the use of a pseudo-random generation subroutine. It is also found that the Exodus method gives a solution as accurate as that obtained with the finite difference method.

References

[1] G. M. Royer, "A Monte Carlo procedure for potential theory of problems," *IEEE Trans. Microwave Theory Tech.*, vol MTT-19, no. 10, pp. 813-818, Oct. 1971.

[2] R. M. Bevensee, "Probabilistic potential theory applied to electrical engineering problems," *Proc. IEEE*, vol. 61, no. 4, pp. 423-437, Apr. 1973.

[3] F. Sanchez-Quesada *et al.*, "Monte Carlo method for dielectric inhomogeneous problems," *Proc. IEE*, vol. 125, no. 12, pp. 1400-1402, Dec. 1978.

[4] R. Schlott, "A Monte Carlo method for the dirichlet problem of dielectric wedges," *IEEE Trans. Microwave Theory Tech.*, vol. 36, no. 4, pp. 724-730, Apr. 1988.

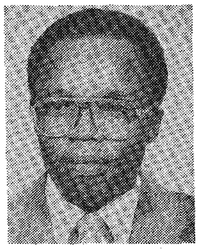
[5] M. N. O. Sadiku, "Monte Carlo solution of axisymmetric potential problems," in *Proc. IEEE Ind. Appl. Soc. Annual Metting*, 1990, pp. 1894-1900.

[6] A. F. Emery and W. W. Carson, "A modification to the Monte Carlo method - The exodus method," *J. Heat Transf.*, Trans. ASME, Series C, vol. 90, pp. 328-332, 1968.

[7] M. N. O. Sadiku, "Monte Carlo methods in an introductory electromagnetic course," *IEEE Trans. Educ.*, vol 33, no. 1, pp. 73-80, Feb. 1990.

[8] M. N. O. Sadiku, *Elements of Electromagnetics*. New York. Holt, Rinehart & Winston, 1989, pp. 671-674

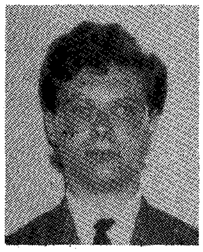
[9] S. Frankel, *Multiconductor Transmission Line Analysis*. Norwood, MA: Artech House, 1977, pp. 359–366.
[10] M. DiStasia and W. C. McHarris, Electrostatic problems? Relax!," *Am. J. Phys.*, vol. 47, no. 5, pp. 440–444, May 1979.
[11] W. H. McCrea and F. J. W. Whipple, "Random paths in two and three dimensions," in *Proc. Roy. Soc. Edinb.*, vol. 60, 1940, pp. 281–298.
[12] S. A. Schelkunoff, *Applied Mathematics for Engineers and Scientists*, 2nd Ed. New York: Van Nostrand, 1965, pp. 465–472.
[13] M. N. O. Sadiku, *Numerical Techniques in Electromagnetics*. Boca Raton, FL: CRC Press, 1992, ch. 2.
[14] H. Kobayashi, *Modeling and Analysis: An Introduction to System Performance Evaluation Methodology*. Reading: MA: Addison-Wesley, 1978, pp. 221–247.



Matthew N. O. Sadiku (S'82–M'82–M'88–SM'90) received the B.Sc. degree in 1978 from Ahmadu Bellow University, Zaria, Nigeria and the M.Sc. and Ph.D. degrees from Tennessee Technological University, Cookeville, TN in 1982 and 1984, respectively.

From 1984 to 1988, he was an Assistant Professor at Florida Atlantic University, where he did graduate work in computer science. Since August, 1988, he has been with Temple University, Philadelphia, PA, where he is presently an Associate Professor. His current research interests are in the areas of numerical techniques in electromagnetics and computer communication networks.

He is the author of over 30 professional papers and four books including *Elements of Electromagnetics* (Saunders, 1989) and *Numerical Techniques in Electromagnetics* (CRC, 1992). He is a Registered Professional engineer and a member of ASEE. He is presently the IEEE Region 2 Student Activities Committee Chairman.



David T. Hunt was born in Spokane, Washington on March 1, 1970. He is currently a senior undergraduate in Electrical Engineering at Temple University, Philadelphia.